Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00024-7
J. Appl. Maths Mechs, Vol. 65, No. 2, pp. 205–217, 2001

0021-8928/01/$—see front matter

MECHANICAL SYSTEMS WITH SERVOCONSTRAINTS†

Yu. F. GOLUBEV

Moscow

(*Received* 9 *June* 2000)

The properties of mechanical systems controlled by the application of generally non-linear servoconstraints, holonomic or non-holonomic, are investigated. The concept of ideal servoconstraints is analysed. The specific features involved in applying the local variational principles of mechanics and the equations of motion derived from them for systems with ideal and non-ideal servoconstraints are pointed out. The problem of a point moving at a constant velocity in a gravity field is solved, on the assumption that the constraint keeping the velocity constant is implemented both by ideal and by non-ideal servocontrols. © 2001 Elsevier Science Ltd. All rights reserved.

The concept of a servoconstraint was apparently first introduced in 1922 by Henri Beghin [1] for systems controlled by servodrives. We will assume that servoconstraints, like traditional mechanical constraints, are given as relations among the coordinates and velocities of the system, and constitute a mathematical formulation of the requirements imposed as the objective of the control. It is assumed that the system is controlled by servomechanisms, whose purpose is to realize the servo-constraints. Unlike traditional, passive constraints, which are presumed to hold precisely at any given time, servoconstraints may be relations of arbitrary structure among the coordinates and velocities of the system and are not necessarily satisfied precisely. All that is required is that the servodrive control should produce a stable transient in which the servoconstraint is an attractor for the phase trajectories of the system. This effect may be achieved by appropriate synthesis of the control of the servoconstraint mismatch.

After the aforementioned synthesis problem has been solved, the equations of the servoconstraints become the same as the traditional equations of constraints in mechanics, except that they may be expressed by arbitrary functions of the coordinates and the velocities. Another difference is that the efforts necessary to realize servoconstraints must guarantee the specially constructed servodrive controls. With this in mind, the problem of investigating the motion of systems with servoconstraints has all the features characteristic for problems of analytical mechanics. The relevant arsenal of methods needs only a few refinements.

Differential constraints which are non-linear in the velocities have been studied before [2]. As yet, the question of whether this kind of constraint [3] can be realized using a passive mechanical device cannot be answered in the affirmative. In this paper we propose to introduce tools of active control that will realize constraints which are arbitrary functions of the states of the system. As is traditional, we will treat a mechanical system as a set of point masses, understanding the reactions of the servoconstraints to be additional forces applied to these point masses so as to realize the servo-constraints.

The local principles of analytical mechanics are based on the concept of ideal constraints, which in turn use the definition of virtual displacement [4]. It has been proposed [3] that the concept of virtual displacement should be extended to the case of non-linear differential constraints given in Lagrangian coordinates, based on the requirement that the D'Alembert–Lagrange principle and Gauss's principle must be consistent. The structure of the set of reactions of non-linear differential constraints has been analysed [5],‡ and the concept of virtual displacement has been extended to the case of non-linear constraints which depend on the radius vectors and velocities of the individual points of the system. The consistency of the generalized concept of virtual displacement with the local variational principles of mechanics has been confirmed. In this paper these aspects of the theory are developed for systems with servoconstraints.

†*Prikl. Mat. Mekh.* Vol. 65, No. 2, pp. 211–224, 2001.

‡See also GOLUBEV, Yu. F., Local variational principles of mechanics for systems with differential non-linear constraints. Preprint No. 54, Inst. Prikl. Mat. im. M. V. Keldysha, Moscow, 1999.

1. TRANSIENT

The restrictions on the motion of a controlled mechanical system may be specified by means of constraints, that is to say, relations – equalities or inequalities – either among the coordinates, or among the coordinates and velocities, of the points in the system. Clearly, these restrictions may not be fulfilled at some given time (such as the starting time), and it is required to construct servodrive controls so as to make the system satisfy the required restrictions and ensure that they will hold throughout the subsequent motion. Let us assume that the mechanical system consists of N point masses in space E^3, whose motion is constrained by both geometrical and differential bilateral constraints. Any geometrical constraint

$$f(\mathbf{r}_1, \ldots, \mathbf{r}_N, t) = 0$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_N$ are the radius vectors of the points and t is the time, may be given the form of a differential constraint

$$\frac{df}{dt} = \sum_{\nu=1}^{N} \frac{\partial f}{\partial \mathbf{r}_\nu} \cdot \mathbf{v}_\nu + \frac{\partial f}{\partial t} = 0$$

where we have put $\dot{\mathbf{r}}_\nu = \mathbf{v}_\nu$. Suppose this operation has been carried out for all geometrical constraints, so that the entire set of traditional mechanical and servoconstraints is taken into account by the following system of independent equations

$$\Phi_j(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N, t) = 0, j = 1, \ldots, m$$

The action exerted by both the servodrives and the mechanical constraints on the νth point mass of the system will be represented by the reaction $\mathbf{R}_\nu$ of the constraint.

The problem of restoring disrupted constraints may be formulated as the problem of constructing a method of control which, because of the equations of motion, realizes the functions†

$$\Phi_j = \tilde{\Phi}_j(t), \quad f_j = \tilde{f}_j(t), \quad j = 1, \ldots, m$$

possessing the following properties.

1. If at some time t_0 we have $\Phi_j(t_0) = 0$, and in the case of a geometrical constraint also $f_j(t_0) = 0$, then for $t > t_0$ necessarily $\tilde{\Phi}_j(t) \equiv 0$, $\tilde{f}_j(t) \equiv 0$ which is analogous to the operation of ordinary mechanical constraints.

2. If servoconstraints are disrupted, the mismatch with these constraints is either asymptotically eliminated at a fairly rapid rate, or eliminated over a finite but fairly short time interval of the transient (piecewise-terminal control).

In particular, in accordance with the operating principle of servomechanisms, it may be required that the time derivatives of the constraints along trajectories of the equations of motion obey the following equations

$$\frac{d\Phi_j}{dt} = \Upsilon_j = \begin{cases} -k_j\Phi_j - \omega_j^2 f_j, & \text{if} \quad \Phi_j = df_j / dt \\ -k_j\Phi_j, & \text{if} \quad \nexists f_j, \quad j = 1, \ldots, m \end{cases}$$

where $k_j < 0$, ω_j^2 are constant force coefficients defining the velocity and quality of the transient and f_j is the left-hand side of the equation of a geometrical constraint in the case when $\Phi_j = df_j/dt$. This method of constructing the transient is the most frequently used in applications [6].

Another type of transient arises if one requires it to take place in minimum time. We will first indicate the synthesis of an optimal control for the case of a geometrical servoconstraint

$$y_1 = f(\mathbf{r}_1, \ldots, \mathbf{r}_N, t), \quad y_2 = \sum_{\nu=1}^{N} \frac{\partial f}{\partial \mathbf{r}_\nu} \cdot \mathbf{v}_\nu + \frac{\partial f}{\partial t}$$

In the (y_1, y_2) plane, the domain of negative values of Υ is situated above the switching curve [7]

$$\tilde{y}_1 = -|y_2| y_2 / 2, \quad -\infty < y_2 < +\infty$$

†GOLUBEV, Yu. F., Dynamics of systems with servoconstraints. Preprint No. 19. Inst. Prikl. Mat. im. M. V. Keldysha, Moscow, 2000.

and the domain of positive values is below the switching curve. There is an easy "prescription" for optimal control. Suppose the point representing the actual state of the system has coordinates (y_1, y_2). The equation of the switching curve may be used to find the corresponding value of $\tilde{y}_1$. The control at the point (y_1, y_2) has the form

$$\Upsilon = \bar{\Upsilon}\cdot\begin{cases} +1 & y_1 < \tilde{y}_1(y_2) \\ -\operatorname{sign} y_2, & y_1 = \tilde{y}_1(y_2) \\ -1, & y_1 > \tilde{y}_1(y_2) \end{cases}$$

where $\bar{\Upsilon}$ is the maximum admissible value of the function Υ. This formula defines an optimal control for the entire phase space.

Given a differential servoconstraint, we have

$$y = \Phi(\mathbf{r}_1,\ldots,\mathbf{r}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N,t)$$

and the solution of the time-optimal problem is given by the relation

$$\Upsilon = \bar{\Upsilon}\cdot\begin{cases} -\operatorname{sign} y, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

Indeed, a control whose output has the maximum absolute value, but with the opposite sign to that of y, guarantees the maximum possible rate of decrease in the function $y(t)$.

Other methods of constructing the transient are possible. For example, taking some parametric family of functions $\Upsilon_i(t)$ $(i = 1, \ldots, m)$, one can designate independent parameters of the family so that the boundary conditions of the transient are satisfied when its duration is fixed [8].

2. REALIZATION OF THE CONSTRAINTS

Taking into account what was stated in the last section, we will henceforth express the equations of the servoconstraint by the formulae

$$f(\mathbf{r}_1,\ldots,\mathbf{r}_N,t) = \tilde{f}(t)$$

for geometrical constraints and

$$\Phi_j(\mathbf{r}_1,\ldots,\mathbf{r}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N,t) = \tilde{\Phi}_j(t), \quad j = 1,\ldots,m$$

for differential constraints, assuming that the functions $\tilde{f}(t)$, $\tilde{\Phi}(t)$ are defined by some mode of control (Section 1).

A necessary and sufficient condition for the deviation of the system from the constraints to obey the above rules at each instant of time by virtue of the equations of motion of the individual point masses is that reactions exist which satisfy the following system of linear equations

$$\sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu}\cdot\frac{\mathbf{R}_\nu}{m_\nu} = b_j, \quad j = 1,\ldots,m, \quad m \leqslant 3N \tag{2.1}$$

where

$$b_j = -\sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu}\cdot\frac{\mathbf{F}_\nu}{m_\nu} - \sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{r}_\nu}\cdot\mathbf{v}_\nu - \frac{\partial \Phi_j}{\partial t} + \Upsilon_j, \quad j = 1,\ldots,m$$

and $\mathbf{F}_\nu$ are the active forces, which differ from those produced by the servodrives. The free terms b_j $(j = 1, \ldots, m)$ are independent of the unknown reactions of the constraints. If $m \leqslant 3N$, the solution of this system is not unique.

We choose an orthonormal frame of reference in the space E^3, say $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. In order to specify the positions of all the point masses in the system (i.e. to define its configuration), it is sufficient to designate the $3N$ scalar coordinates of the system in the configuration space.

Following an approach proposed in [5] (see also the references cited in the footnotes), in order to determine the structure of the set of solutions of system (2.1), we define the following vectors in the configuration space

$$\mathbf{x} = (x_1, \ldots, x_{3N}), \quad \boldsymbol{\delta} = (\delta_1, \ldots, \delta_{3N}), \quad \mathbf{a}_j = (a_{j1}, \ldots, a_{j,3N}) \tag{2.2}$$

where the components are

$$x_i = \frac{\mathbf{R}_\nu \cdot \mathbf{e}_k}{\sqrt{m_\nu}}, \quad \delta_i = \sqrt{m_\nu}\delta\mathbf{r}_\nu \cdot \mathbf{e}_k, \quad a_{ji} = \frac{1}{\sqrt{m_\nu}} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu} \cdot \mathbf{e}_k$$

$$i = k + 3(\nu - 1), \quad k = 1, 2, 3, \quad \nu = 1, \ldots, N, \quad j = 1, \ldots, m$$

and the Euclidean scalar product is defined by

$$(\mathbf{a}_j, \mathbf{x}) = \sum_{i=1}^{3N} a_{ji} x_i$$

The system of equations (2.1) for the unknown vector $\mathbf{x}$ may be written as follows

$$(\mathbf{a}_j, \mathbf{x}) = b_j, \quad j = 1, \ldots, m$$

Since the initial system of constraints is independent, the rank of the matrix of this system is m.

The solution $\mathbf{x}$ may be sought in the form

$$\mathbf{x} = \sum_{k=1}^{m} \lambda_k \mathbf{a}_k + \mathbf{x}_\tau$$

where the vector $\mathbf{x}_\tau$ is orthogonal to all the vectors $\mathbf{a}_j$. The constants λ_k are uniquely defined, since the matrix of the system of equations for λ_k is the Gram matrix of the sequence of linearly independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$. The vector $\mathbf{x}_\tau$ in the equations of the system of constraints cannot be defined, and if it is chosen arbitrarily, the equations of the constraints are not violated during the motion of the material system.

Definition 1. The *normal space* of a system of differential constraints is defined to be a set $\mathfrak{R}$ of sequences $\{\mathbf{R}_\nu, \nu = 1, \ldots, N\}$ of reaction vectors such that the $3N$-dimensional vector $\mathbf{x}$ corresponding to each sequence belongs to the linear span $\text{lin}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$.

Definition 2. The space $\mathcal{T}$ of *virtual displacements* is the set $\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\}$ of sequences of displacement vectors satisfying the system of equations

$$\sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu} \cdot \delta\mathbf{r}_\nu = 0, \quad j = 1, \ldots, m$$

The dimension of the space $\mathcal{T}$ is $3N - m$. A sequence $\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\}$ of vectors satisfying this system of homogeneous linear equations is a *virtual displacement* of the system of point masses.

The vector δ of the virtual displacement $\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\}$ in (2.2) is orthogonal to all the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$:

$$\sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu} \cdot \delta\mathbf{r}_\nu = (\mathbf{a}_j, \boldsymbol{\delta}) = 0, \quad j = 1, \ldots, m$$

The condition that an actual displacement $\{d\mathbf{r}_\nu, \mathbf{v}_\nu dt, \nu = 1, \ldots, N\}$ of the system of point masses should belong to the space of virtual displacements $\mathcal{T}$ is

$$\sum_{\nu=1}^{N} \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu} \cdot \mathbf{v}_\nu = 0, \quad j = 1, \ldots, m$$

The following theorem has been proved (see the footnote on page 205).

Theorem 1. A sequence of reactions $\{\mathbf{R}_\nu, \nu = 1, \ldots, N\}$ will belong to the normal space $\mathcal{R}$ of a system of differential constraints if and only if, for any $\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\} \in \mathcal{T}$,

$$\sum_{\nu=1}^{N} \mathbf{R}_\nu \cdot \delta\mathbf{r}_\nu = 0, \quad \forall\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\} \in \mathcal{T} \tag{2.3}$$

Definition 3. Constraints (mechanical and servo) imposed on a system of N point masses are said to be *ideal* if they satisfy condition (2.3).

In other words, for ideal constraints, and only for them, the reactions belong to the normal space $\mathcal{R}$.

In order to find the reactions of ideal constraints, one can use the method of Lagrange multipliers

$$\mathbf{R}_\nu = \sum_{j=1}^{m} \lambda_j \frac{\partial \Phi_j}{\partial \mathbf{v}_\nu}$$

Each term in the sum on the right of this expression may obviously be interpreted as the reaction of the jth constraint acting on the νth point mass.

Definition 4. A servodrive control that produces reactions of ideal servoconstraints will be called an *ideal servocontrol.*

The concept of an ideal servocontrol means only that the reactions will assuredly belong to the normal space; it is not implied that no energy is consumed by the servodrives.

The requirement that the servoconstraints be ideal may prove to be inconsistent with the design of the servodrives used in a specific problem. However, this does not always mean that such drives can never realize the desired constraints. The simple condition $\mathbf{x}_\tau = 0$ will sometimes prove to be too strong.

The complete set of solutions of the problem is obtained if one adds the non-zero components of the reactions for which the corresponding vector $\mathbf{x}_\tau$ is orthogonal to all the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$. Denote this set of additional reactions by $\{\mathbf{R}_\nu^\tau, \nu = 1, \ldots, N\}$. In order to augment the set of solutions in the required manner, one can take a suitable vector $\delta \perp \operatorname{lin}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ and define $\mathbf{x}_\tau = \beta\delta$ or

$$\mathbf{R}_\nu^\tau = \beta m_\nu \overline{\delta\mathbf{r}}_\nu, \quad \nu = 1, \ldots, N$$

where $\{\overline{\delta\mathbf{r}}_\nu, \nu = 1, \ldots, N\} \in \mathcal{T}$ is an appropriate virtual displacement of the system and β is a dimensional scalar coefficient, generally depending on the state parameters of the system and the time.

3. THE EQUATIONS OF MOTION

It has been shown (see the footnote on page 206) that if the constraints satisfy the condition for a generalized ideal constraint (Definition 3) and the condition for an ideal servocontrol (Definition 4), then the principle of virtual displacements, the D'Alembert-Lagrange principle, and Gauss's principle remain valid. For non-ideal servoconstraints, the D'Alembert-Lagrange principle may obviously be written in the form

$$\sum_{\nu=1}^{N} (m_\nu \mathbf{w}_\nu - \mathbf{F}_\nu - \beta m_\nu \overline{\delta\mathbf{r}}_\nu)\delta\mathbf{r}_\nu = 0, \quad \forall\{\delta\mathbf{r}_\nu, \nu = 1, \ldots, N\} \in \mathcal{T}$$

where $\mathbf{w}_\nu$ are the actual accelerations of the points of the system.

The concept of quasi-coordinates for servoconstraints is practically the same as the similar concept introduced previously (see footnote 1) for non-linear differential constraints. Allowance is made for the fact that the mechanical system may be subject to the action of both ordinary mechanical constraints and servoconstraints. As established in Section 2, the only difference between the equations of these types of constraint is whether the corresponding right-hand sides $\tilde{f}(t)$ (for geometrical constraints) or $\tilde{\Phi}(t)$ (for differential constraints) vanish or not.

Suppose the configuration of a system of N^3 point masses in R, considered with all (mechanical and servo) geometrical constraints, is uniquely defined by the coordinates

$$q_1, \ldots, q_n, \; n \leqslant 3N$$

so that the radius vectors of all the points in the system are expressed by the functions

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q_1, \ldots, q_n, t), \ \nu = 1, \ldots, N$$

Let us assume that the system is subject to differential constraints

$$\Phi_j(\mathbf{r}_1,\ldots,\mathbf{r}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N,t) = \bar{\Phi}(t), \ \ j = 1,\ldots,m \leq n.$$

Noting that

$$\mathbf{v}_\nu = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}_\nu}{\partial t}$$

let us substitute these expressions into the constraint equations. Then the constraints become

$$\phi_j(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_n,t) = 0, \ \ j = 1,\ldots,m$$

Let us assume that the rank of the Jacobian

$$\frac{\partial(\phi_1,\ldots,\phi_m)}{\partial(\dot{q}_1,\ldots,\dot{q}_n)}$$

is m. This means that the system of constraints defines in the velocity space $\dot{q}_1, \ldots, \dot{q}_n$ a surface of dimension $n - m$, which may be represented in parametric form

$$\dot{q}_i = \dot{q}_i(q_1,\ldots,q_n,\dot{\pi}_1,\ldots,\dot{\pi}_{n-m}), \ \ i = 1,\ldots,n$$

in such a way that, when the scalar parameters $\dot{\pi}_1, \ldots, \dot{\pi}_{n-m}$, – called quasi-velocities – are specified arbitrarily, the equations of the differential constraints are automatically satisfied.

If desired, the functions $\dot{\pi}_k(t)$ may be regarded as the derivatives of the quasi-coordinates $\pi_k(t)$.

The question of when quasi-coordinates may be considered as completely legitimate coordinates of a system of point masses has been answered (see the footnote on page 205).

Let us take the standard definition of the partial derivative of the coordinate q_i with respect to the quasi-coordinate π_k

$$\frac{\partial q_i}{\partial \pi_k} = \frac{\partial \dot{q}_i}{\partial \dot{\pi}_k}$$

Similarly, for an arbitrary function $f(q_1, \ldots, q_n, t)$, we obtain

$$\frac{\partial f}{\partial \pi_k} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial \pi_k} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{\pi}_k}$$

Using quasi-coordinates in the case of servoconstraints, we can define the space of virtual displacements as before:

$$\delta\mathbf{r}_\nu = \sum_{i=1}^{n} \frac{\partial \mathbf{v}_\nu}{\partial \dot{q}_i}\delta q_i = \sum_{i=1}^{n} \frac{\partial \mathbf{r}_\nu}{\partial q_i}\delta q_i, \ \ \delta q_i = \sum_{k=1}^{n-m} \frac{\partial \dot{q}_i}{\partial \dot{\pi}_k}\delta\pi_k = \sum_{k=1}^{n-m} \frac{\partial q_i}{\partial \pi_k}\delta\pi_k$$

Suppose that, corresponding to the coordinates $q_1, \ldots, q_n$, we have generalized forces $Q_1, \ldots, Q_n$. The scalar quantity

$$Q_k^* = \sum_{i=1}^{n} Q_i \frac{\partial q_i}{\partial \pi_k}$$

is a generalized force corresponding to the quasi-coordinate π_k.

The concept of ideal servoconstraints enables one to separate the problem of investigating the dynamical properties of a system from that of determining the forces produced by the servodrives, and to derive equations of motion in which the reactions of the ideal servoconstraints do not occur. For non-linear ideal differential servoconstraints, one can apply the generalized Voronets equations (see the footnote on page 205). Here we will further generalize the Voronets equations to the case of non-ideal servoconstraints.

We reduce the differential constraints to the form

$$\dot{q}_{p+\nu} = \varphi_{p+\nu}(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_p), \quad p = n - m, \quad \nu = 1, \ldots, m$$

and form the functions

$$T^* = T^*(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_p, t) \quad \varphi^*_{p+\nu}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_p, t)$$

The function T^* is obtained from the kinetic energy T of the system by replacing the velocities $\dot{q}_{p+\nu}(\nu = 1, \ldots, m)$ by their expressions in terms of the differential constraints, while the functions $\varphi^*_{p+\nu}$ are identical in form with the corresponding functions $\varphi_{p+\nu}$. However, the derivatives of the functions T^* and $\varphi^*_{p+\nu}$ are evaluated using the rule for differentiation with respect to a quasi-coordinate

$$\frac{\partial q_{p+\nu}}{\partial q_k} = \frac{\partial \dot{q}_{p+\nu}}{\partial \dot{q}_k} = \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_k}, \quad \nu = 1, \ldots, m, \quad k = 1, \ldots, p$$

The partial derivatives of the functions T and $\varphi_{p+\nu}$ will be evaluated, as before, as if all their arguments were independent. Then, proceeding as before (see footnote 1), taking into account the components of the reactions not satisfying the condition for ideal constraints, we obtain the following theorem.

Theorem 2. The coordinates q_i define the motion of a mechanical system obeying the differential servoconstraints if and only if they satisfy the following system of equations

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_i}\right) - \frac{\partial T^*}{\partial q_i} = Q_i^* + Q_i^\tau + \tilde{Q}_i, \quad i = 1, \ldots, p = n - m$$

where

$$Q_i^* = Q_i + \sum_{\nu=1}^{m} Q_{p+\nu} \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i} = Q_i + \sum_{\nu=1}^{m} Q_{p+\nu} \frac{\partial q_{p+\nu}}{\partial q_i}$$

$$Q_i^\tau = \overline{Q}_i + \sum_{\nu=1}^{m} \overline{Q}_{p+\nu} \frac{\partial q_{p+\nu}}{\partial q_i}, \quad \overline{Q}_j = \beta \sum_{\nu=1}^{N} m_\nu \overline{\delta \mathbf{r}}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_j}, \quad j = 1, \ldots, n$$

$$\tilde{Q}_i = \sum_{\nu=1}^{m} \frac{\partial T}{\partial \dot{q}_{p+\nu}} \left[\frac{d}{dt}\left(\frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}\right) - \frac{\partial \varphi^*_{p+\nu}}{\partial q_i}\right]$$

and it is assumed that the necessary sequence of reactions $\{\mathbf{R}_\nu, \nu = 1, \ldots, N\}$ is realized at each instant of time.

Remark 1. The equations system of Theorem 2 holds for any constraints imposed on the system. But it is not complete. To complete it, it suffices to add the equations of the differential constraints (kinematic equations) and to choose a virtual displacement of the system $\{\overline{\delta \mathbf{r}}_\nu, \nu = 1, \ldots, N\}$ in a suitable manner.

Remark 2. The terms $\tilde{Q}_i$ are due to the action of the differential constraints. In the general case, they may involve the second derivatives of the generalized coordinates, so they cannot always be treated as generalized forces. At the same time, in the case of linear differential constraints, the operator in square brackets in the formulae for $\tilde{Q}_i$ does not involve the second derivatives of the coordinates, and $\tilde{Q}_i$ may then be interpreted as certain generalized forces that arise owing to the action of non-holonomic constraints [9].

Let us find the sum

$$\sum_{i=1}^{p} \tilde{Q}_i \dot{q}_i = \sum_{\nu=1}^{m} \frac{\partial T}{\partial \dot{q}_{p+\nu}} \left[\sum_{i=1}^{p} \frac{d}{dt}\left(\frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}\right)\dot{q}_i - \sum_{i=1}^{p} \frac{\partial \varphi^*_{p+\nu}}{\partial q_i}\dot{q}_i\right] =$$

$$= \sum_{\nu=1}^{m} \frac{\partial T}{\partial \dot{q}_{p+\nu}} \left[\frac{d}{dt}\left(\sum_{i=1}^{p} \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}\dot{q}_i\right) - \sum_{i=1}^{p} \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}\ddot{q}_i - \sum_{i=1}^{p} \frac{\partial \varphi_{p+\nu}}{\partial q_i}\dot{q}_i - \right.$$

$$-\sum_{\mu=1}^{m}\sum_{i=1}^{p}\frac{\partial\varphi_{p+\nu}}{\partial q_{p+\mu}}\frac{\partial\varphi_{p+\mu}}{\partial\dot q_i}\dot q_i\Bigg]=\sum_{\nu=1}^{m}\frac{\partial T}{\partial\dot q_{p+\nu}}\left[\frac{d}{dt}\left(\sum_{i=1}^{p}\frac{\partial\varphi_{p+\nu}}{\partial\dot q_i}\dot q_i-\varphi_{p+\nu}\right)+\right.$$

$$\left.+\sum_{\mu=1}^{m}\frac{\partial\varphi_{p+\nu}}{\partial q_{p+\mu}}\left(\varphi_{p+\mu}-\sum_{i=1}^{p}\frac{\partial\varphi_{p+\mu}}{\partial\dot q_i}\dot q_i\right)+\frac{\partial\varphi_{p+\nu}}{\partial t}\right]$$

Lemma 1. If the differential constraints are such that the actual displacements belong to the set of virtual displacements, then

$$\sum_{i=1}^{p}\tilde Q_i\dot q_i=\sum_{\nu=1}^{m}\frac{\partial T}{\partial\dot q_{p+\nu}}\frac{\partial\varphi_{p+\nu}}{\partial t}$$

Proof. The fact that the actual displacements belong to the set of virtual displacements means that

$$\dot q_{p+\nu}=\sum_{i=1}^{p}\frac{\partial\varphi_{p+\nu}}{\partial\dot q_i}\dot q_i\rightarrow\sum_{i=1}^{p}\frac{\partial\varphi_{p+\nu}}{\partial\dot q_i}\dot q_i=\varphi_{p+\nu},\quad \nu=1,\dots,m$$

Corollary 1. If the differential constraints are stationary and allow the actual displacements to belong to the set of virtual displacements, then

$$\sum_{i=1}^{p}\tilde Q_i\dot q_i=0$$

4. THE MOTION OF A POINT AT A VELOCITY OF CONSTANT MAGNITUDE

Let us imagine a robot which is required to paint a vertical wall using a paint spray. The wall is to be painted uniformly, so that the motion of the robot's arm must be such that the spray moves at a velocity of constant magnitude. Another practical problem, leading to the requirement that the gripping device of a manipulator must move at a velocity of constant magnitude, is encountered when polishing surfaces [6].

Let us consider the simplest possible mechanical system (see the footnote on page 206) in which a point of mass m (the manipulator's gripping device) can move in a vertical plane and has two degrees of mobility. One degree changes the distance ρ from the point m to a given fixed point O in the same plane; the other modifies the angle between the vertical and the segment connecting the points m and O. Servodrives ensure motion in each degree of mobility. The mass of the manipulator arm together with the servodrives will be ignored. It is required to construct a motion in which, under the action of gravity and the servodrives, the magnitude of the velocity of the point is kept fixed. Incidentally, the motion of a point at a constant velocity in a central field was considered in [2].

We will use the theory proposed above. We form expressions for the kinetic energy and force function

$$T=\frac{m}{2}(\dot\rho^2+\rho^2\dot\varphi^2),\quad U=mg\rho\cos\varphi$$

where g is the acceleration due to gravity and φ is the polar angle between the direction of the force of gravity and the ray from O to m. The servoconstraint is

$$\dot\rho^2+\rho^2\dot\varphi^2-v_0^2=\tilde\Phi(t)$$

On the assumption that the servodrives produce an ideal control, the Lagrange equations with multiplier λ have the form

$$\ddot\rho=\rho\dot\varphi^2+g\cos\varphi+\lambda\dot\rho,\quad \rho^2\ddot\varphi=-2\rho\dot\rho\dot\varphi-g\rho\sin\varphi+\lambda\rho^2\dot\varphi$$

In order to find λ, differentiate the equation of the constraint and replace the second derivatives by their expressions from the equations of motion. This gives

$$\lambda = \frac{\Upsilon - 2g(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi)}{2(v_0^2 + \tilde\Phi)}$$

Thus, an ideal control may be obtained if we require the servodrive responsible for the linear degree of mobility ρ to produce a force $u_\rho = \dot\rho\lambda$ per unit transferred mass, and the servodrive responsible for the rotational degree of mobility φ to produce a torque $u_\varphi = \rho^2\dot\varphi\lambda$ per unit mass. Motion with this control will obey the generalized equation of Theorem 2 if we assume that $Q_i^\tau = 0$. Successful investigation of the equation depends essentially on the choice of the system of coordinates.

Let us take Cartesian coordinates

$$y = \rho\sin\varphi, \quad z = -\rho\cos\varphi$$

The equation of the differential constraint may be represented in the form

$$\dot y^2 + \dot z^2 = v_0^2 + \tilde\Phi, \quad \text{or} \quad \dot z = \pm\sqrt{v_0^2 + \tilde\Phi - \dot y^2}$$

where a positive value of $\dot z$ corresponds to upward motion of the manipulator arm and a negative value to downward motion.

Let us consider the case $\dot z > 0$. Then

$$\dot z = \varphi^* = \sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}, \quad \frac{\partial z}{\partial y} = \frac{\partial \dot z}{\partial \dot y} = -\frac{\dot y}{\sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}}$$

The functions T^* and U^* are take the form

$$T^* = \frac{m}{2}[v_0^2 + \tilde\Phi(t)], \quad U^* = -mgz$$

The function T^* depends neither on the velocities nor on the coordinates.

We have the following expressions for Q_y^* and $\tilde Q_y$:

$$Q_y^* = \frac{\partial U^*}{\partial z}\frac{\partial z}{\partial y} = -mg\frac{\partial \dot z}{\partial \dot y} = mg\frac{\dot y}{\sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}}$$

$$\tilde Q_y = -m\dot z\frac{d}{dt}\left(\frac{\dot y}{\sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}}\right)$$

Consequently, we obtain a system of equations that does not involve the reactions of the ideal servo-constraints

$$\dot z\frac{d}{dt}\left(\frac{\dot y}{\dot z}\right) = g\frac{\dot y}{\dot z}, \quad \dot z = \sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2} \tag{4.1}$$

This system has the obvious solution

$$y \equiv y_0, \quad z = \int_{t_0}^{t} \sqrt{v_0^2 + \tilde\Phi(t)}\, dt + z_0$$

meaning that the gripping device goes through the starting position (y_0, z_0) and moves vertically upward at a velocity close to the prescribed velocity.

Now consider the case $\dot z < 0$

$$\dot z = \varphi^* = -\sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}, \quad \frac{\partial z}{\partial y} = \frac{\partial \dot z}{\partial \dot y} = \frac{\dot y}{\sqrt{v_0^2 + \tilde\Phi(t) - \dot y^2}}$$

The dynamical equation of motion takes the same form (4.1) as obtained for positive $\dot z$. In the case

$\dot{z} < 0$ we have the obvious solution

$$y \equiv y_0, \quad z = -\int_{t_0}^{t} \sqrt{v_0^2 + \tilde{\Phi}(t)}\, dt + z_0$$

meaning that the gripping device goes through the starting position (y_0, z_0) and moves vertically downward at a velocity close to the prescribed velocity.

Let us find solutions for which $\dot{y} \neq 0$. We introduce the variable

$$z' = \begin{cases} \sqrt{v_0^2 + \tilde{\Phi}(t) - \dot{y}^2} / \dot{y}, & \dot{z} \geqslant 0 \\ -\sqrt{v_0^2 + \tilde{\Phi}(t) - \dot{y}^2} / \dot{y}, & \dot{z} < 0 \end{cases}$$

equal to the tangent of the angle of inclination of the tangent to the trajectory. Then

$$\dot{y}^2 = \frac{1}{1 + z'^2}(v_0^2 + \tilde{\Phi}(t)), \quad \dot{z}^2 = \frac{z'^2}{1 + z'^2}(v_0^2 + \tilde{\Phi}(t))$$

and there is no finite value of z' for which $\dot{y}$ vanishes. To fix our ideas, let us take $\dot{y} > 0$.

The equation for the variable z' is the same whether it takes positive or negative values:

$$\frac{dz'}{\sqrt{1 + z'^2}} = -\frac{g dt}{\sqrt{v_0^2 + \tilde{\Phi}(t)}}$$

We put

$$\eta = \eta(t) = c \exp\left(-\int_{t_0}^{t} \frac{g dt}{\sqrt{v_0^2 + \tilde{\Phi}(t)}} \right) \to 0 \quad \text{as} \quad t \to \infty$$

where $c = z'(t_0) + \sqrt{1 + [z'(t_0)]^2}$ is a constant. The solution of the differential equation for z' may be expressed in the form

$$z' + \sqrt{1 + z'^2} = \eta, \quad z' = -(1 - \eta^2)/(2\eta)$$

Consequently, $z' \to -\infty$ as $t \to +\infty$. It can be shown that

$$\begin{aligned} &c > 1, && \text{if} \quad z'(t_0) > 0 \\ &c = 1, && \text{if} \quad z'(t_0) = 0 \\ &0 < c < 1, && \text{if} \quad z'(t_0) < 0 \end{aligned}$$

Thus, if, say, $z'(t_0) > 0$, then the gripping device of the manipulator will first move upward, reaching a maximum point, and then turn downward at a velocity of constant magnitude. If $z'(t_0) < 0$, the trajectory of the grip will not include an upward section. The portrait of the motion for the case $\dot{y} < 0$ is similar.

Suppose, as before, that $\dot{y} > 0$ (the investigation of the case $\dot{y} < 0$ is analogous). Consider the function $y(t)$ when $\dot{y} > 0$. It may be constructed by integrating the equation

$$\dot{y} = \frac{2\eta}{1 + \eta^2}\sqrt{v_0^2 + \tilde{\Phi}(t)}$$

We see that since $\eta > 0$, the right-hand side of this equation is positive. Let us estimate it. Obviously

$$\dot{y} \leqslant 2\eta\sqrt{v_0^2 + \tilde{\Phi}(t)} \leqslant 2\eta\sqrt{v_0^2 + |\tilde{\Phi}(t)|} \leqslant a \exp(-\varkappa[t - t_0])$$

$$a = 2c\sqrt{v_0^2 + |\tilde{\Phi}(t_0)|}, \quad \varkappa = \frac{g}{\sqrt{v_0^2 - |\tilde{\Phi}(t_0)|}}$$

Integrating, we obtain

$$y - y_0 \leq \frac{a}{\varkappa}\{1 - \exp(-\varkappa[t - t_0])\} < \frac{a}{\varkappa}$$

Consequently, the variable y tends to some limit value $\bar{y}$, determined by the initial data. At the same time, by the rule for the choice of signs, we have

$$\dot{z} = -\frac{1-\eta^2}{1+\eta^2}\sqrt{v_0^2 + \bar{\Phi}(t)}$$

and for sufficiently large t the vertical velocity is negative and of magnitude close to v_0. At the same time, the trajectory of the motion approaches the vertical asymptote $y = \bar{y}$.

Let us assume now that the degree of mobility with respect to the coordinate φ is free, only the degree of mobility in the coordinate ρ being governed by a servodrive. Then, obviously, an ideal servoconstraint cannot be realized; but this does not mean that no control can be chosen that realizes the servoconstraint (though the latter need not be ideal). In order to solve the problem, we have to assume that the constraint reaction has the form

$$\mathbf{R} = \lambda\mathbf{v} + \mathbf{R}^\tau, \quad \mathbf{R}^\tau \cdot \mathbf{v} = 0$$

The second equality means that the vector $\mathbf{R}^\tau$ is orthogonal to the normal space of the system. As before, the multiplier λ is determined from the condition that the constraint holds, and the vector $\mathbf{R}^\tau$ is determined from the condition that the reaction $\mathbf{R}$ is collinear with the radius vector $\mathbf{r}$:

$$\lambda = m\frac{g\dot{z} + \Upsilon}{v_0^2 + \bar{\Phi}}, \quad \mathbf{R}^\tau = \lambda\left(\frac{v^2}{\mathbf{v}\cdot\mathbf{r}}\mathbf{r} - \mathbf{v}\right), \quad \mathbf{R} = m\frac{g\dot{z} + \Upsilon}{\mathbf{v}\cdot\mathbf{r}}\mathbf{r}$$

The force $\mathbf{R}$ produced by the slave mechanism of the servodrive will be able to guarantee constant velocity if the quantity $(\mathbf{v}\cdot\mathbf{r})$ is not too small. Note that this was not the case for an ideal servocontrol.

Consider steady motion, when $\bar{\Phi}(t) = \Upsilon = 0$. In Cartesian coordinates $y = \rho\sin\varphi$, $z = -\rho\cos\varphi$, the equations of motion become

$$\ddot{y} = 2g\dot{z}y\left[\frac{d(\rho^2)}{dt}\right]^{-1}, \quad \ddot{z} = -g + 2g\dot{z}z\left[\frac{d(\rho^2)}{dt}\right]^{-1}, \quad \rho^2 = y^2 + z^2$$

This system of equations may also be written in the form

$$\frac{\ddot{y}}{y} = \frac{\ddot{z}+g}{z} = 2g\dot{z}\left[\frac{d(\rho^2)}{dt}\right]^{-1}$$

As might have been expected, it admits of motion along the vertical straight line $y \equiv 0$ at $\dot{z} = \pm v_0$.

Let us find a solution for which the trajectory is a circle of radius R

$$\left(z + \frac{g}{\omega^2}\right)^2 = y^2 = R^2 \Rightarrow \rho^2 = R^2 - \frac{g^2}{\omega^4} - \frac{2g}{\omega^2}z \Rightarrow \frac{d(\rho^2)}{dt} = -\frac{2g}{\omega^2}\dot{z}$$

Then the system of equations of motion takes the form

$$\ddot{y} + \omega^2 y = 0, \quad \ddot{z} + \omega^2 z = -g$$

We will write down the desired solution

$$y = a\cos\omega t + b\sin\omega t, \quad z = b\cos\omega t - a\sin\omega t - \frac{g}{\omega^2}, \quad a^2 + b^2 = R^2$$

The velocity vector has components

$$\dot{y} = -a\omega\sin\omega t + b\omega\cos\omega t, \quad \dot{z} = -b\omega\sin\omega t - a\omega\cos\omega t$$

so that the motion will take place at a constant velocity $\upsilon_0 = \omega R$.

At a fixed velocity υ_0, the circular trajectories form a family depending on the radius R. Each circle has its centre at the point with coordinates

$$y_c = 0, \quad z_c = -gR^2/\upsilon_0^2$$

and intersects the Oz axis at two points: the upper point with ordinate z_M and the lower point with ordinate z_m:

$$z_M = R - \frac{gR^2}{\upsilon_0^2}, \quad z_m = -\left(R + \frac{gR^2}{\upsilon_0^2}\right)$$

The function $z_M(R)$ has a maximum $\bar{z}_M$ when $R = \upsilon_0^2/(2g)$

$$\bar{z}_M = \max_R z_M(R) = \frac{\upsilon_0^2}{4g}$$

As R increases from zero to $\upsilon_0^2/(2g)$, each successive circle contains any preceding circle inside it. As R increases from $\upsilon_0^2/(2g)$ to infinity, the circles intersect, and the coordinate z_m decreases monotonically. The projection of the phase domain of the solutions we have constructed on to the coordinate space is the union of the upper half-disk, with radius $r = \upsilon_0^2/(2g)$ and centre at the point $(0, -\upsilon_0^2/(4g)$, and the domain defined by the relations.

$$z = -\frac{gR^2}{\upsilon_0^2}, \quad -R \leqslant y \leqslant R \quad \text{for} \quad R \geqslant \frac{\upsilon_0^2}{2g}$$

This family of solutions obtained shows that the way in which the mode of realization of a servoconstraint has a crucial effect on the nature of the motion. Thus, a point moving at constant velocity in a gravity field may describe a circle if the constraint is realized by a purely radial force; but it cannot have circular trajectories when the servodrives develop an ideal control.

5. CONCLUSION

The main results of this paper are as follows.

1. The concept of virtual displacements for systems of point masses obeying servoconstraints has been generalized.
2. An analytical representation of the reactions of the servoconstraints, depending on the structure of the virtual displacements, has been given.
3. A concept of quasi-coordinates for non-linear differential servoconstraints has been developed, and the Voronets equations have been extended to the case of differential servoconstraints non-linear in the velocities.
4. Two problems on the motion of a heavy point with the magnitude of the velocity kept constant have been solved. One problem was solved on the assumption that the differential constraint was realized by an ideal servocontrol and the other on the assumption that the servocontrol was realized only in the radial direction. It has been shown that the trajectory of the point depends essentially on the selected mode of servocontrol.

This research was supported financially by the Russian Foundation for Basic Research (98-01-00065, 98-01-00805) and the Federal Special-Purpose "Integration" Programme (1.6-329, A0097).

REFERENCES

1. APPELL, P., *Traité de Mécanique Rationelle*, Vol. 2, *Dynamique des Systèmes*. Mécanique Analytique. Gauthier-Villars, Paris, 1950.
2. DOBRONRAVOV, V. V., *Elements of the Mechanics of Non-Holonomic Systems*. Vysshaya Shkola, Moscow, 1970.

3. CHETAYEV, N. G., *Stability of Motion. Papers on Analytical Mechanics*. Izd. Akad. Nauk SSSR, Moscow, 1962.
4. GOLUBEV, Yu. F., *Elements of Theoretical Mechanics*, 2nd ed., revised and enlarged. Izd. Mosk. Gos. Univ., Moscow, 2000.
5. POLYAKHOV, N. N., ZEGZHDA, S. A. and YUSHKOV, M. P., The Suslov-Jourain principle as a corollary of the equations of dynamics. In: *Collection of Scientific-Methodological Articles on Theoretical Mechanics*, No. 12. Vysshaya Shkola, Moscow, 1982, pp. 72—79.
6. NAKANO, E., *Introduction to Robot Engineering*. Mir, Moscow, 1988.
7. PONTRYAGIN, L. S., BOLTYANSKII, V. G., GAMKRELIDZE, R. V. and MISHCHENKO, Ye F., *Mathematical Theory of Optimal Processes*. Fizmatgiz, Moscow, 1961.
8. OKHOTSIMSKII, D. Ye., GOLUBEV, Yu. F. and SIKHARULIDZE, Yu. G., Algorithms for *Controlling Spacecraft on Entering the Atmosphere*. Nauka, Moscow, 1975.
9. RUMYANTSEV, V. V., On Hamilton's principle for non-holonomic systems. *Prikl. Mat. Mekh*. 1978, **42**, 3, 387—399.

Translated by D.L.